Reinforcement Learning

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Lecture 8: Alignment and Reasoning with Reinforcement Learning

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License Information for Reinforcement Learning (EE-568)

A paradigm shift in Large Language Models (LLMs)

- Reasoning the process that makes an LLM solve tasks that are unsolvable by just next token prediction .
- An example is the results on math benchmarks.

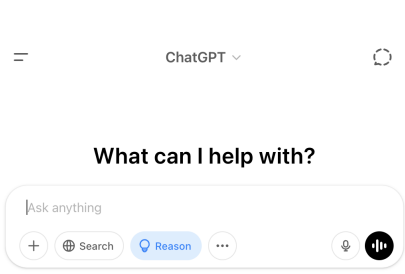
AIME Benchmark

Model (17)	Accuracy	Cost In / Out	Latency (s)
1 o3 Mini	86.5%	$1.10 / $4.40	154.65 s
2 DeepSeek R1	74.0%	$8.00 / $8.00	153.91 s
3 o1	71.5%	$15.00 / $60.00	177.03 s
4 Claude 3.7 Sonnet (Thinking)	52.7%	$3.00 / $15.00	303.71 s
5 Gemini 2.0 Flash (001)	29.8%	$0.10 / $0.40	11.21 s
6 Claude 3.7 Sonnet	22.5%	$3.00 / $15.00	18.93 s
7 Gemini 1.5 Pro (002)	18.7%	$1.25 / $5.00	10.64 s
8 Gemini 1.5 Flash (002)	17.3%	$0.07 / $0.30	5.70 s
9 Llama 3.3 Instruct Turbo (70B)	16.9%	$0.88 / $0.88	11.24 s
10 Grok 2	15.2%	$2.00 / $10.00	57.88 s
11 GPT 4o (2024-08-06)	14.0%	$2.50 / $10.00	68.37 s
12 GPT 4o (2024-11-20)	12.1%	$2.50 / $10.00	15.78 s
13 GPT 4o Mini	11.5%	$0.15 / $0.60	28.77 s
14 Claude 3.5 Sonnet Latest	10.0%	$3.00 / $15.00	9.19 s
15 Mistral Large (11/2024)	9.6%	$2.00 / $6.00	19.68 s
16 Mistral Small (02/2024)	5.6%	$0.20 / $0.60	13.23 s
17 Claude 3.5 Haiku Latest	3.5%	$1.00 / $5.00	9.05 s

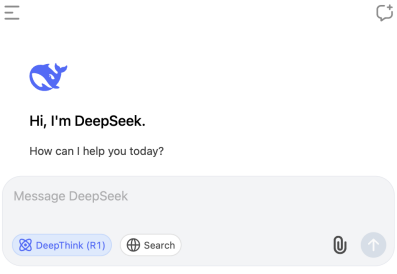
https://www.vals.ai/benchmarks/aime-2025-03-11

Reasoning chatbots

- More and more LLMs include the *reasoning* option.



https://chatgpt.com/



https://chat.deepseek.com/

- Next, we look at possible ways to turn a next token prediction model to a reasoning model.
- **The current "reasoning" paradigm is based on RL!**

From Chain-of-Thought to Reasoning

- With Chain-of-Thought (CoT) [18], we give the model examples of how to reason.

Multiple Q/A examples

CoT examples:

Q: Roger has 5 tennis balls. He buys 2 more cans of tennis balls. Each can has 3 tennis balls. How many tennis balls does he have now?

A: Roger started with 5 balls. 2 cans of 3 tennis balls each is 6 tennis balls. $5 + 6 = 11$. The answer is 11.

Q: $\cdots$ **A:** $\cdots$

User Prompt: s_i

Q: The cafeteria had 23 apples. If they used 20 to make lunch and bought 6 more, how many apples do they have?

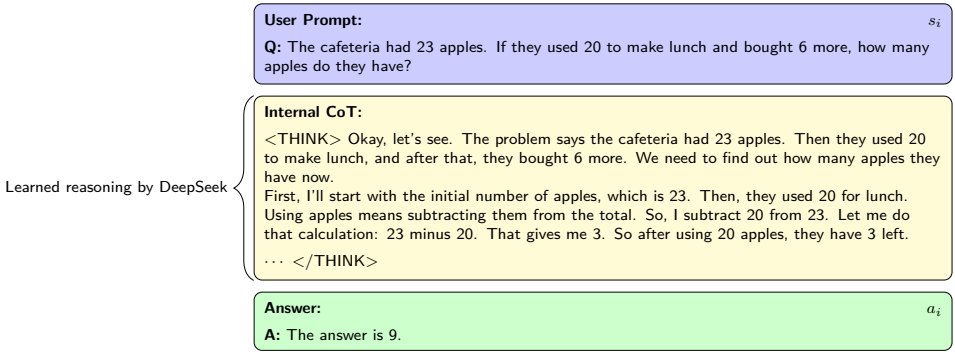
Answer: a_i

A: The cafeteria had 23 apples originally. They used 20 to make lunch. So they had $23 - 20 = 3$. They bought 6 more apples, so they have $3 + 6 = 9$. The answer is 9.

- When given a new question, we expect the model to reason similarly to the examples above.
- The model first plans, then it answers.

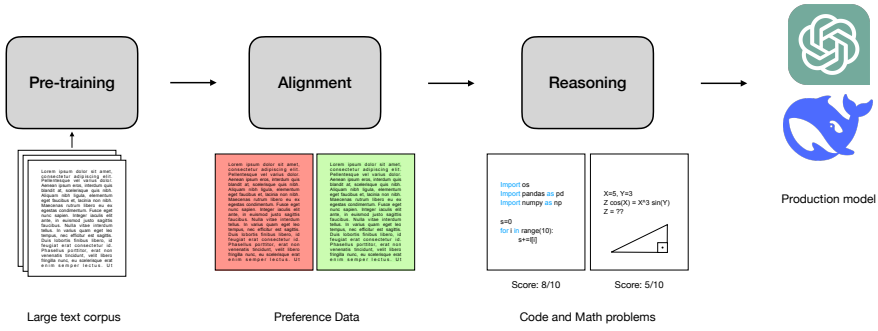
From Chain-of-Thought to Reasoning

- With Reasoning, we aim at **teaching the model how to reason from rewards**.



- The internal CoT is not copied from examples, not prompted, but learned from "reward" maximization.

Outline



- This lecture
  1. Basics of language models
  2. Fundamentals of pre-training
  3. Reinforcement learning with human feedback (RLHF)
  4. Reasoning with reinforcement learning

A motivation for language models (LMs)

Example

Predict the next word w given the following source sentence S_{source}?

S_{source} : *"On January 1 people usually say happy new [w]."*

A motivation for language models (LMs)

Example

Predict the next word w given the following source sentence S_{source}?

S_{source} : *"On January 1 people usually say happy new [w]."*

Question: ○ Why is this important?

- spelling & grammar correction — $\mathsf{p}(\text{year}|S_{\text{source}}) > \mathsf{p}(\text{years}|S_{\text{source}})$
- machine translation — $\mathsf{p}(S_{\text{translation 1}}|S_{\text{source}}) > \mathsf{p}(S_{\text{translation 2}}|S_{\text{source}})$
- sentence classification — $\mathsf{p}(S_{\text{class 1}}|S_{\text{source}}) > \mathsf{p}(S_{\text{class 2}}|S_{\text{source}})$
- speech recognition — $\mathsf{p}(w|S_{\text{source}})$
- chatbot — $\mathsf{p}(w|S_{\text{source}})$
- (more generally) labeling, automated decisions,...

Basics for language models (LMs) – I

Definition (Language model [7])

Models that assign probabilities to sequences of words are called language models.

Remarks:

- Given a sentence with T words: $S = w_{1:T} = (w_1, \ldots, w_T)$, by the chain rule of probability:

$$\mathsf{p}(S) = \mathsf{p}(w_{1:T}) = \mathsf{p}(w_1)\mathsf{p}(w_2|w_1)\mathsf{p}(w_3|w_{1:2}) \cdots \mathsf{p}(w_T|w_{1:T-1}) = \prod_{t=1}^{T} \mathsf{p}(w_t|w_{1:t-1})$$

- Implicitly, we are enforcing a graphical model that takes "time" into account.

Example

If $S = w_{1:3}$ = "happy new year", then $\mathsf{p}(S) = \mathsf{p}(\text{happy})\mathsf{p}(\text{new}|\text{happy})\mathsf{p}(\text{year}|\text{happy new})$.

Basics for language models (LMs) – II

Question:

- How can we compute $\mathsf{p}(w_t|w_{1:t-1})$?

Remarks:

- A trivial solution: Just count the frequency on a large corpus, e.g.,

$$\mathsf{p}(\text{year}|S_{\text{source}}) = \frac{\mathsf{p}(S_{\text{source}} + \text{year})}{\mathsf{p}(S_{\text{source}})} \approx \frac{\#(\text{On January 1 people usually say happy new year})}{\#(\text{On January 1 people usually say happy new})}$$

- But the language is creative, there are several ways to express the same meaning.
- The sentence above might even not appear on the corpus.
- We need better ways to estimate such probabilities!

N-gram LMs

Markov assumption [9]

The probability of a word only depends on the last $N-1$ words as

$$\mathrm{p}(w_t|w_{1:t-1}) = \mathrm{p}(w_t|w_{t-N:t-1}) \approx \frac{\#(w_{t-N:t})}{\#(w_{t-N:t-1})}.$$

Figure: Markov in 1913 [9] used "Markov chains" to predict whether the upcoming letter would be a vowel or a consonant.

Example

In the bigram LM ($N=2$), we only need to estimate $\mathrm{p}(w_t|w_{t-1}) \approx \frac{\#(w_{t-1:t})}{\#(w_{t-1})}$ to generate text.

w_t

w_{t-1}	i	want	to	eat
i	5	827	0	9
want	2	0	608	1
to	2	0	4	686
eat	0	0	2	0

w_{t-1}

i	want	to	eat
2533	927	2417	746

$\rightarrow$

w_t

w_{t-1}	i	want	to	eat
i	0.002	0.33	0	0.0036
want	0.0022	0	0.66	0.0011
to	0.00083	0	0.0017	0.28
eat	0	0	0.0027	0

Figure: Count (Left) and probability $\mathrm{p}(w_t|w_{1:t-1})$ (Right) from the Berkeley Restaurant Project corpus of 9332 sentences [7].

Towards pre-training an N-gram LM

- In natural language processing (NLP), we use tokens to represent words coming from a vocabulary $\mathcal{V}$.

Terminologies:

- A *token* is the smallest unit that can be assigned a meaning to be processed.
  - In English, a token often corresponds to a word.
  - However, a single token can also encode compound words like *New York*.
  - In Chinese or Japanese, there is no space between words.
  - In these languages, sentence segmentation is required before we tokenize.
- We indicate the beginning and the end of sentences with tokens $\langle \mathrm{BOS} \rangle$ and $\langle \mathrm{EOS} \rangle$.
  - S_{source} "$\langle \mathrm{BOS} \rangle$ Happy new year $\langle \mathrm{EOS} \rangle$" has $T = 5$ tokens.
- The size of our vocabulary is denoted as $|\mathcal{V}|$.
- *Pre-training*: building a LM based on a large corpus in a (often) self-supervised manner.
- *Inference*: Using a trained LM to do next word prediction.

N-gram LMs: "Pre-training" & Inference

- The following simplified examples show the difficulty of pre-training and inference with 2-gram LMs.

"Pre-training"
1. Count $\#(w_{t-1})$ and $\#(w_{t-1:t})$ over the corpus.
2. Obtain probability $\mathsf{p}(w_t \mid w_{t-1})$ over the corpus.

Inference
1. Set w_1 as $\langle \mathrm{BOS} \rangle$, $t = 1$.
2. **While True**:
▸ $w_{t+1} = \arg\max_{w \in \mathcal{V}} \mathsf{p}(w \mid w_t)$
▸ **If** w_{t+1} is $\langle \mathrm{EOS} \rangle$: **break**
▸ $t = t + 1$
3. Output: $[w_1, \cdots, w_{t+1}]$.

Remarks:

- Need to store the probability for all N-gram pairs.
- Language is creative, some new N-gram pairs might not even appear on the corpus.
- Cannot incorporate earlier words than N due to the Markov assumption.

$$\mathsf{p}(\text{two} \mid \text{one plus one equals}) = \mathsf{p}(\text{two} \mid \text{it is wrong that one plus one equals})?$$

The optimization objective

- A (vector-output) neural network $\mathbf{h}_\theta \in \Delta^{|\mathcal{V}|-1}$ can be used to model such probability.

$$-\log \mathsf{p}_\theta(\mathbf{b}_{1:T}) = -\log\left(\prod_{t=1}^{T} \mathsf{p}_\theta(\mathbf{b}_t|\mathbf{b}_{1:t-1})\right) = \sum_{t=1}^{T}\left(-\log \underbrace{\mathsf{p}_\theta(\mathbf{b}_t|\mathbf{b}_{1:t-1})}_{\mathbf{h}_\theta(\mathbf{b}_{1:t-1})^{[\text{“}\mathbf{b}_t\text{”}]}}\right)$$

$$= \sum_{t=1}^{T}\left(-\log \mathbf{h}_\theta(\mathbf{b}_{1:t-1})^{[\text{“}\mathbf{b}_t\text{”}]}\right) = \sum_{t=1}^{T}\left(-\sum_{i=1}^{|\mathcal{V}|} \hat{\mathbf{u}}_t^{[i]} \log \mathbf{u}_t^{[i]}\right) = \text{cross entropy loss}$$

- $\mathbf{u}_t := \mathbf{h}_\theta(\mathbf{b}_{1:t-1}) \in \mathbb{R}^{|\mathcal{V}|}$ is the probability distribution of the next word given previous $t-1$ words.
- $\hat{\mathbf{u}}_t \in \mathbb{R}^{|\mathcal{V}|}$ is the correct distribution (one-hot) at t step.

Remarks:

- Teacher forcing training: We always give the model the correct history sequence.
- Auto-regressive inference: The history sequence comes from its prediction result.
- Notation: We will use s for prompts and a for an answer sampled from $\pi := \mathbf{h}_\theta$ in the sequel.

Alignment: going beyond next token predictions

- After the training, we need methods to enforce certain behaviours of the LLM.

Examples:
- Impose that the LLM follows instructions.
- Avoid bad words in the LLM responses.
- Make it likely that the LLM outputs sentences that match human taste.

- Alignment can be performed considering different inputs.
  - Demonstrations
  - Demonstrations + online access to the environment
  - Preferences
  - Preferences + online access to the environment
  - Reward function

Alignment from demonstrations

- We are given a dataset $\mathcal{D}_{\text{demo}}$ of prompts $\{s_i\}_{i=1}^N$ and desired answers $\{a_i\}_{i=1}^N$.
- We aim at maximizing the likelihood of the answers in the dataset.
- This can be done via imitation learning (see Lecture 7).

Alignment via Behavioral Cloning

1: The learner receives: (i) A dataset $\mathcal{D}_{\text{demo}} = \{(s_i, a_i)\}_{i=1}^{N_{\text{demo}}}$, (ii) A policy function parameter $\theta \in \Theta$.
2: The learner computes the loss: $\widehat{\ell}_{\text{BC}}(\theta) := \sum_{i=1}^{N_{\text{demo}}} \left[-\log \pi_\theta \left(a_i | s_i \right) \right]$.
3: The learner outputs $\pi_{\theta^\star}$ with $\theta^\star = \arg\min_{\theta \in \Theta} \widehat{\ell}_{\text{BC}}(\theta)$.

Alignment from demonstrations + online access

- A common method for this setting is SPIN (Self-Play fIne-tuNing) [3], which considers a two player game:
  - One player tries to generate answers as similar as possible as the observed ones in $\mathcal{D}_{\text{demo}}$.
  - The other player aims at distinguish answers of the first player from the the ones in $\mathcal{D}_{\text{demo}}$.
- At iteration t, both "players" aim at solving the following bilevel formulation:

$$\theta_{t+1} = \arg\max_{\theta \in \Theta} \mathbb{E}_{s\sim\rho, a\sim\pi(\cdot|s;\theta_t)} \left[r_{t+1}(s,a) - \eta^{-1} D_{\text{KL}}(\pi(\cdot|s;\theta)||\pi(\cdot|s;\theta_t)) \right], \tag{1}$$

$$\text{s.t.} \quad r_{t+1} = \arg\min_{r \in \mathcal{R}} \mathbb{E}_{s,a\sim\mathcal{D}_{\text{demo}}} \mathbb{E}_{a'\sim\pi(\cdot|s;\theta_t)} \left[\ell(r(s,a) - r(s,a')) \right], \tag{2}$$

  - where $\ell(\cdot)$ is a monotonically decreasing, non negative, smooth and convex function,
  - ρ is the prompt distribution,
  - η is the step size.
- The choice $\ell(x) = -x$ corresponding to minimize an integral probability metric.
- A common choice in practice is the logistic loss function $\ell(s) = \log\left(1 + \exp(-s)\right)$.

Alignment from demonstrations + online access (continued)

- The upper level has an explicit solution $\pi(a|s;\theta_{t+1}) \propto \pi(a|s;\theta_t)\exp\left(\eta r_{t+1}(s,a)\right)$.
  - Question: What if $\theta_{t+1} \notin \Theta$?
- Assuming $\theta_{t+1} \in \Theta$, the both problems reduce to a single optimization problem:

$$\theta_{t+1} = \arg\max_{\theta\in\Theta} \mathbb{E}_{s,a\sim\mathcal{D}_{\text{demo}}} \mathbb{E}_{a'\sim\pi(\cdot|s;\theta_t)} \left[-\ell\left(\eta^{-1}\log\left(\frac{\pi(a|s;\theta)}{\pi(a|s;\theta_t)}\right) - \eta^{-1}\log\left(\frac{\pi(a'|s;\theta)}{\pi(a'|s;\theta_t)}\right)\right)\right]. \quad (3)$$

- **Self-study:** Compare it with DPO to be introduced in few slides.

Alignment from preferences: Setup

- A preference dataset is formed by
  - Prompts $\{s_i\}_{i=1}^N$.
  - Response pairs $\left\{a_i^+, a_i^-\right\}_{i=1}^N$.
- a_i^+ denote the preferred response to the prompt s_i according to the preference model p_{pref}.
- The preference model is a mapping as follows $\mathsf{p}_{\text{pref}} : \mathcal{A} \times \mathcal{A} \to [0, 1]$.
- $\mathsf{p}_{\text{pref}}(a, a'|s)$ is the probability that a is preferred to a' in response to s.
- We start focusing on a specific preference model known as Bradley-Terry model [2].

Bradley-Terry model

For some unknown reward function r, the preferences are generated as follows

$$\mathsf{p}_{\text{pref}}(a, a'|s) = \sigma(r(s, a) - r(s, a')) = \frac{1}{1 + e^{r(s,a')/r(s,a)}},$$

where σ is the sigmoid function.

Alignment from preferences via RLHF

- The most popular learning from preferences paradigm is RLHF (RL from Human Feedback).
- RLHF follows a two step procedure.
- First, we fit a reward model $\widehat{r}$ on the preference dataset.
- Second, we learn a policy parameterization $\theta^\star$ that
  - maximizes $\widehat{r}$;
  - does not drift excessively from the pretrained model π_{ref} as measured by $\beta D_{\text{KL}}(\pi(\cdot|s;\theta^\star)||\pi_{\text{ref}}(\cdot|s))$
- All in all, we aim at solving the following bilevel problem

$$\theta^\star = \arg\max_{\theta\in\Theta} \mathbb{E}_{s\sim\rho, a\sim\pi(\cdot|s)} \left[\widehat{r}(s,a;\phi)\right] - \beta\mathbb{E}_{s\sim\rho}[D_{\text{KL}}(\pi(\cdot|s;\theta)\,\|\pi_{\text{ref}}(\cdot|s))]. \tag{4}$$

$$\text{s.t.} \quad \widehat{r} = \arg\max_{r\in\mathcal{R}} \sum_{i=1}^{N} \log\sigma(r(s_i,a_i^+) - r(s_i,a_i^-)). \tag{5}$$

- β is called the "alignment parameter."
- For larger β, the solution will be more *aligned* to the reference model π_{ref}.

The RLHF algorithm

- The two steps in pseudocode look as follows.

RLHF (Reinforcement Learning from Human Feedback)

1: The learner receives: (i) A preference dataset $\mathcal{D}_{\text{pref}} = \{(s_i, a_i^+, a_i^-)\}_{i=1}^{N_{\text{pre}}}$, (ii) A prompt dataset $\mathcal{D}_{\text{prompts}} = \{s_i\}_{i=1}^{N_{\text{prompts}}}$, (iii) A reward function class $\mathcal{R}$, (iv) A policy function parameter class Θ.

2: The learner estimates the reward as $r(\cdot, \cdot; \phi^\star)$ (typically an NN with parameters ϕ) via

$$\widehat{r} = \arg\max_{r \in \mathcal{R}} \sum_{i=1}^{N} \log\left(\sigma\Big(r(s_i, a_i^+; \phi) - r(s_i, a_i^-; \phi)\Big)\right). \qquad \text{(Reward Loss)}$$

3: The learner then approximately solves the following problem:

$$\theta^\star \in \arg\max_{\theta \in \Theta} \mathbb{E}_{s\sim\rho, a\sim\pi(\cdot|s)}\left[\widehat{r}(s, a; \phi)\right] - \beta \mathbb{E}_{s\sim\rho}[D_{\text{KL}}(\pi(\cdot|s; \theta) \,\|\pi_{\text{ref}}(\cdot|s))], \qquad \text{(RLHF)}$$

using PPO or REINFORCE. Note that we do not have all the ingredients to solve the problem.

- Thereoretical improvements can be obtained replacing the KL with the sum of KL and χ^2 divergences [4].

Learning from preferences without online access

- RLHF requires to generate new responses to the prompt dataset.
- It is more computationally efficient to leverage only the preference dataset.
- DPO (Direct Preference Optimization) [14] was introduced towards this goal.

Direct Preference Optimization

1: The learner receives as input:
   - A preference dataset $\mathcal{D}_{\text{pref}} = \{(s^i, a_i^+, a_i^-)\}_{i=1}^N$.
   - A policy function parameters class Θ.

2: The learner computes the stochastic loss.

$$\widehat{\ell}_{\text{DPO}}(\theta) := -\frac{1}{|\mathcal{D}_{\text{pref}}|} \sum_{s,a^+,a^- \in \mathcal{D}_{\text{pref}}} \left[\log \left(\sigma \left(\beta \log \left(\frac{\pi(a^+|s;\theta)}{\pi_{\text{ref}}(a^+|s)} \right) - \beta \log \left(\frac{\pi^\star(a^-|s;\theta)}{\pi_{\text{ref}}(a^-|s)} \right) \right) \right) \right]$$

3: The learner outputs $\pi_{\theta^\star}$ with $\theta^\star = \arg\min_{\theta \in \Theta} \widehat{\ell}_{\text{DPO}}(\theta)$.

DPO derivation (Part 1)

- For a fixed $r(\cdot, \cdot; \phi)$, the solution to (RLHF) (for expressive enough policy classes!) is given by

$$\pi^\star_\phi(a|s) = \frac{\pi_{\text{ref}}(a|s) \exp(\beta^{-1} r(s, a; \phi))}{Z(s, \phi, \beta)}, \qquad \text{(Optimal Policy)}$$

  where we defined $Z(s, \phi, \beta) := \sum_{a'} \pi_{\text{ref}}(a'|s) \exp(\beta^{-1} r(s, a'; \phi))$.

- By rearranging, it holds that for all s, a that $\log(\pi^\star_\phi(a|s)) = \log(\pi_{\text{ref}}(a|s)) + \beta^{-1} r(s, a; \phi) - \log(Z(s, \phi, \beta))$.

- Such a quantity cannot be computed in closed form because computing $Z(s, \phi, \beta)$ is intractable.

- For two possible answers a, a' to the *same* question s, it holds that

$$\log(\pi^\star_\phi(a|s)) - \log(\pi^\star_\phi(a'|s)) = \log(\pi_{\text{ref}}(a|s)) - \log(\pi_{\text{ref}}(a'|s)) + \beta^{-1} r(s, a; \phi) - \beta^{-1} r(s, a'; \phi).$$

DPO derivation (Part 2)

- Therefore, the reward functions difference can be computed as

$$r(s,a;\phi) - r(s,a';\phi) = \beta \log \left(\frac{\pi^\star_\phi(a|s)}{\pi_{\text{ref}}(a|s)} \right) - \beta \log \left(\frac{\pi^\star_\phi(a'|s)}{\pi_{\text{ref}}(a'|s)} \right). \tag{6}$$

- The computation is efficient because the normalization constant $Z(s,\phi,\beta)$ does not appear.
- It follows that we can plug in the above analytical solution into (Reward Loss)

$$\begin{aligned} \min_{\phi\in\Phi} \ \ell(\phi) &:= -\mathbb{E}_{s\sim\rho,(a^-\prec a^+)\sim \mathsf{p}_{\text{pref}}(\cdot|s)} \left[\log \left(\sigma \Big(\beta \log \left(\frac{\pi^\star_\phi(a^+|s)}{\pi_{\text{ref}}(a^+|s)} \right) - \beta \log \left(\frac{\pi^\star_\phi(a^-|s)}{\pi_{\text{ref}}(a^-|s)} \right) \Big) \right) \right] \\ \text{s.t.} \quad \pi^\star_\phi &:= \arg\max_{\pi\in\Pi} \ \mathbb{E}_{s\sim\rho,a\sim\pi(\cdot|s)} \left[r(s,a;\phi) \right] - \beta \mathbb{E}_{s\sim\rho}[D_{\text{KL}}(\pi(\cdot|s) \,\|\pi_{\text{ref}}(\cdot|s))]. \end{aligned}$$

- The bilevel problem is still too complicated, DPO [14] is derived to ignore the lower level problem.

$$\arg\min_{\theta\in\Theta} \ \ell_{\text{DPO}}(\theta) := -\mathbb{E}_{s\sim\rho,(a^-\prec a^+)\sim \mathsf{p}_{\text{pref}}(\cdot|s)} \left[\log \left(\sigma \left(\beta \log \left(\frac{\pi(a^+|s;\theta)}{\pi_{\text{ref}}(a^+|s)} \right) - \beta \log \left(\frac{\pi(a^-|s;\theta)}{\pi_{\text{ref}}(a^-|s)} \right) \right) \right) \right]. \tag{7}$$

DPO vs RLHF

- DPO is easier computationally. It uses only one neural network.
- DPO and RLHF are not equivalent because we dropped the constraint on the policy in the last step.
- Therefore, the solution set of the DPO optimization problem includes the RLHF solution set [20].
- Some DPO solutions assign high probability to answers unseen in the preference dataset.
- To fix the above issue one can constrain the probability mass moved [1].
- For the above reason, DPO requires using early stopping in practice.
- DPO can be extended in the multi stage setting [13].

Some limitations of the Bradley-Terry model

- The Bradley-Terry model can only capture transitive preferences.
- Averaging across humans might not give a dataset where transitivity holds.
- As an example, let us consider 3 humans h_1, h_2, h_3 and 3 possible answers y_1, y_2, y_3.
- Let us denote $\mathsf{p}^h_{\text{pref}}$ the preference model of human h defined as follows

$$\begin{aligned}
&\mathsf{p}^{h_1}_{\text{pref}}(y_1, y_2) = 1 \quad \mathsf{p}^{h_1}_{\text{pref}}(y_2, y_3) = 0 \quad \mathsf{p}^{h_1}_{\text{pref}}(y_3, y_1) = 1 \\
&\mathsf{p}^{h_2}_{\text{pref}}(y_1, y_2) = 0 \quad \mathsf{p}^{h_2}_{\text{pref}}(y_2, y_3) = 1 \quad \mathsf{p}^{h_2}_{\text{pref}}(y_3, y_1) = 1 \\
&\mathsf{p}^{h_3}_{\text{pref}}(y_1, y_2) = 1 \quad \mathsf{p}^{h_3}_{\text{pref}}(y_2, y_3) = 1 \quad \mathsf{p}^{h_3}_{\text{pref}}(y_3, y_1) = 0.
\end{aligned}$$

- Each of these models is transitive.
- However, the average model defined as $\mathsf{p}_{\text{pref}}(y, y') = \frac{1}{3} \sum_{h \in \{h_1, h_2, h_3\}} \mathsf{p}^h_{\text{pref}}(y, y')$ satisfies

$$\mathsf{p}_{\text{pref}}(y_1, y_2) = \mathsf{p}_{\text{pref}}(y_2, y_3) = \mathsf{p}_{\text{pref}}(y_3, y_1) = 2/3.$$

- That is, the average model is non transitive and can not be modeled by the BT assumption.

Nash learning from human feedback (NLHF)

- NLHF allows to use general (possibly non transitive) preference models.

Nash Learning from Human Feedback [11]

1: The learner estimates the preference model as $\mathsf{p}_{\mathrm{pref}}(\cdot,\cdot|\cdot;\phi^\star)$ with

$$\phi^\star = \arg\min_{\phi\in\Phi} \widehat{\ell}_{\mathrm{PM}}(\phi) := -\sum_{i=1}^{N} \log\left(\mathsf{p}_{\mathrm{pref}}(a_i^+, a_i^- | s^i;\phi)\right).$$

2: Sample $a^i \sim \pi(\cdot|s^i;\theta)$ and $a'^{,i} \sim \pi(\cdot|s^i;\theta')$ for all $i \in [N]$.

3: The learner computes the stochastic objective

$$\widehat{\ell}_{\mathrm{NLHF}}(\theta,\theta') := -\frac{1}{|\mathcal{D}_{\mathrm{prompts}}|}\sum_{i=1}^{N} \mathsf{p}_{\mathrm{pref}}(a^i, a'^{,i}|s^i;\phi^\star) + \beta D_{\mathrm{KL}}(\pi\left(\cdot|s^i;\theta\right)\|\pi_{\mathrm{ref}}\left(\cdot|s^i\right)) - \beta D_{\mathrm{KL}}(\pi\left(\cdot|s^i;\theta'\right)\|\pi_{\mathrm{ref}}\left(\cdot|s^i\right)),$$

4: The learner outputs $\pi_{\theta^\star}$ with $\theta^\star = \arg\min_{\theta\in\Theta}\max_{\theta'\in\Theta}\widehat{\ell}_{\mathrm{NLHF}}(\theta,\theta')$.

Finding a saddle point of $\widehat{\ell}_{\mathrm{NLHF}}(\theta, \theta')$

- From a computational perspective general preferences are harder.
- The reason is that we now need to solve a minmax problem.
- Notice that under the Bradley-Terry assumption a minimization was enough.
- Simple solving with gradient descent ascent has two problem:
  - No last iterate guarantees.
  - Slow $\mathcal{O}(1/\sqrt{T})$ rate.
- In the next slides we see how to overcome these two challenges.

Strongly convex, strongly concave case: Nash MD

- For $\beta > 0$ the problem is strongly convex-strongly concave.
- Assuming above and a **single state**, [11] showed that Nash-MD has $\mathcal{O}(1/T)$ last iterate convergence.

Nash-MD

1: $\pi_1 = \pi_{\text{ref}}$, learning rate η.
2: **for** $t = 1, \ldots, T$ **do**
3: Compute mixture between initial policy and π_t

$$\bar{\pi}_t(a) = \frac{\pi_t(a)^{1-\beta\eta}\pi_{\text{ref}}(a)^{\beta\eta}}{\sum_{b\in\mathcal{A}} \pi_t(b)^{1-\beta\eta}\pi_{\text{ref}}(b)^{\beta\eta}}.$$

4: Compute the averaged preference model

$$\mathsf{p}^{\bar{\pi}_t}_{\text{pref}}(a) = \sum_{b\in\mathcal{A}} \bar{\pi}_t(b)\mathsf{p}_{\text{pref}}(a,b).$$

5: Mirror descent step with gradient evaluated in $\bar{\pi}_t$: $\pi_{t+1}(a) = \arg\min_{\pi\in\Pi} \left[\langle \pi, \mathsf{p}^{\bar{\pi}_t}_{\text{pref}}\rangle + \frac{1}{\eta} D_{\text{KL}}(\pi, \bar{\pi}_t)\right]$.
6: **end for**

- Is Nash MD applying gradient descent ascent on $\widehat{\ell}_{\text{NLHF}}(\theta, \theta')$?

Gradient descent ascent

- Almost! Find the differences in red.

Gradient Descent Ascent

1: $\pi_1 = \pi_{\text{ref}}$, learning rate η.
2: **for** $t = 1, \dots, T$ **do**
3: Compute mixture between initial policy and π_t

$$\bar{\pi}_t(a) = \frac{\pi_t(a)^{1-\beta\eta}\pi_{\text{ref}}(a)^{\beta\eta}}{\sum_{b\in\mathcal{A}} \pi_t(b)^{1-\beta\eta}\pi_{\text{ref}}(b)^{\beta\eta}}.$$

4: Compute the averaged preference model

$$\mathsf{p}_{\text{pref}}^{\pi_t}(a) = \sum_{b\in\mathcal{A}} \pi_t(b)\mathsf{p}_{\text{pref}}(a,b).$$

5: Mirror descent step with gradient evaluated in π_t: $\pi_{t+1}(a) = \arg\min_{\pi\in\Pi} \left[\langle \pi, \mathsf{p}_{\text{pref}}^{\pi_t}\rangle + \frac{1}{\eta} D_{\text{KL}}(\pi, \bar{\pi}_t)\right]$.
6: **end for**

- Since the game is antisymmetric the two players produces exactly the same iterates.
- That's why we talk about gradient descent ascent but it is enough to keep only one sequence of iterates.

Multi Turn Preference Optimization (MTPO)

- [15] uses Gradient Descent Ascent in the multi state setting deriving MTPO.

MTPO

1: $\pi_1 = \pi_{\text{ref}}$, learning rate η.
2: **for** $t = 1, \dots, T$ **do**
3: Let $Q^{\pi,\pi'}$ denote the state action value functions associated to th preference model p_{pref}.
4: Apply the gradient descent ascent at each state s using $Q^{\pi,\pi'}$:

- $\bar{\pi}_t(a|s) = \frac{\pi_t(a)^{1-\beta\eta}\pi_{\text{ref}}(a)^{\beta\eta}}{\sum_{b\in\mathcal{A}} \pi_t(b)^{1-\beta\eta}\pi_{\text{ref}}(b)^{\beta\eta}}$.
- $\underline{Q}^{\pi_t,\pi_t}(s,a) = \sum_{b\in\mathcal{A}} \pi_t(b) Q^{\pi_t,\pi_t}(s,a,b)$.
- $\pi_{t+1}(\cdot|s) = \arg\min_{\pi\in\Pi} \left[\langle \pi(\cdot|s), \underline{Q}^{\pi_t,\pi_t}(s,\cdot)\rangle + \frac{1}{\eta} D_{\text{KL}}(\pi(\cdot|s), \bar{\pi}_t(\cdot|s))\right]$.

5: **end for**

- MTPO can be seen to converge at a $\mathcal{O}(1/T)$ rate in the strongly convex concave setting.
- In the convex concave setting instead gradient descent ascent and therefore MTPO can divergence.

Optimism to fix it

- Let us revisit optimistic gradient descent.
- Let Δ denote the probability simplex.

Optimistic gradient descent ascent (OGDA)

1: Initialize x_1 uniformely.
2: **for** $t = 1, \dots, T$ **do**
3: Update the decision avariable x_{t+1} as follows:

$$x_{t+1} = \arg\min_{x \in \Delta} \left[\langle x, 2\nabla f(x_t) - \nabla f(x_{t-1}) \rangle + \frac{1}{\eta} D_{\mathrm{KL}}(x, x_t) \right]$$

.
4: **end for**

- The average iterate of ODGA converges at $\mathcal{O}(1/T)$ for convex concave games.
- Asymptotically the last iterate converges strictly faster.
- The rate becomes $o(1/t)$ for all t large enough.
- [19] applies this technique to obtain faster convergence rate in NLHF.

Learning to reason with RL

User Prompt: s_i

"A rectangle's length is three times its width. If the perimeter of the rectangle is 64 units, what are its dimensions?"

Internal Chain of Though (CoT): b_i

<THINK>
Step 1: Let the width be s. Then the length is $3x$.

Step 2: The perimeter is given by $P = 2(s + 3x) = 8x$. So, $8x = 64$ and $s = 8$.

Step 3: Calculate the length: $3x = 3 \times 8 = 24$.

</THINK>

Final Answer: a_i

<ANSWER> The rectangle has a width of 8 units and a length of 24 units. </ANSWER> r_i

- **Question:** How can we learn the internal CoT from questions s_i and rewards r_i?

Group relative policy optimization (GRPO)

- **Problem:** The PPO algorithm requires estimating a value function, which is as big as our LLM.

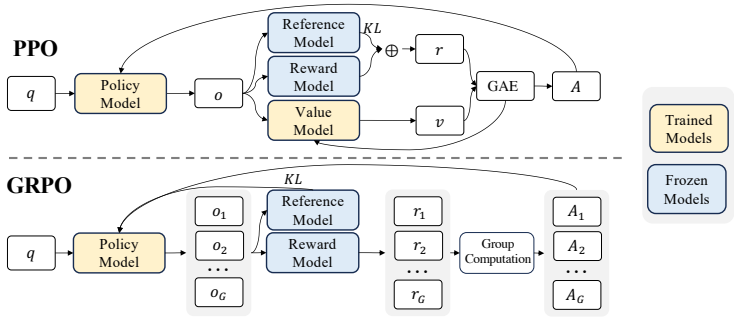


Figure: Source: DeepSeekMath `https://arxiv.org/abs/2402.03300`

- **Solution:** Estimate A by sampling several answers and computing relative rewards within the sample.
- Previous work had similar ideas for removing the value model dependency [8].

Group relative policy optimization (GRPO)

GRPO (optimization formulation) [16]

$$\max_{\theta} \quad \mathbb{E}_{s'\sim\lambda_{\mu}^{\pi_{\theta_t}},\{a_i\}_{i=1}^{G}\sim\pi_{\theta_t}(\cdot|s)} \frac{1}{G}\sum_{i=1}^{G}\min\left\{\frac{\pi_\theta(a_i|s)}{\pi_{\theta_t}(a_i|s)}A_i^{\pi_{\theta_t}}(s,a_i), \text{clip}\left(\frac{\pi_\theta(a_i|s)}{\pi_{\theta_t}(a_i|s)}; 1-\epsilon; 1+\epsilon\right)A_i^{\pi_{\theta_t}}(s,a)\right\}$$
$$- \beta \mathrm{KL}(\pi_\theta(\cdot|s)||\pi_{\text{ref}}(\cdot|s))$$

Remarks:

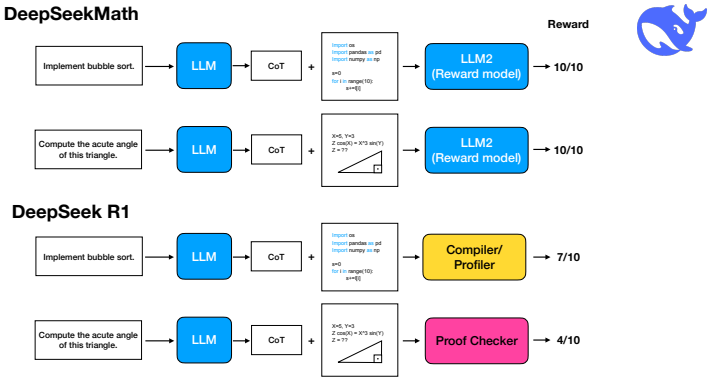
- The advantages are estimated as $A_i^{\pi_{\theta_t}}(s,a_i) = \frac{r(s,a_i) - G^{-1}\sum_{i=1}^{G} r(s,a_i)}{\sqrt{G^{-1}\sum_{i=1}^{G}\left(r(s,a_i) - G^{-1}\sum_{i=1}^{G} r(s,a_i)\right)^2}}$.
- Here G represents the number of actions sampled as a group at each state, i.e., its group size.
- GRPO uses the following unbiased estimator [6] to estimate the KL divergence:

$$\mathrm{KL}(\pi_\theta(\cdot|s)||\pi_{\text{ref}}(\cdot|s)) = \frac{\pi_{\text{ref}}(a_i|s)}{\pi_\theta(a_i|s)} - \log\frac{\pi_{\text{ref}}(a_i|s)}{\pi_\theta(a_i|s)} - 1$$

- By computing group relative rewards, GRPO reduces update variance and ensures stable learning.

DeepSeek R1: training on raw rewards

- In the DeepSeekMath paper, the rewards still come from a reward model.
- **Problem:** The reward model is not perfect and overoptimizing can lead to *reward hacking*.



- **Solution:** We can query code compiler/interpreters and proof checkers to get high quality rewards.

Other approaches

- OpenAI were the first to use RL for reasoning, with no details disclosed [12].
- Kimi 1.5 uses an off-policy ℓ_2 regularized policy gradient method with the same baseline as GRPO [17].
- Many details about the reward model of DeepSeekR1 remain undisclosed.
- HuggingFace is organizing an open source reproduction of DeepSeekR1: OpenR1 [5].
- **Question:** is RL really needed?

Simple reasoning without RL

- Through simple sampling tricks, we can replicate the reasoning behaviour of reasoning LLMs [10]

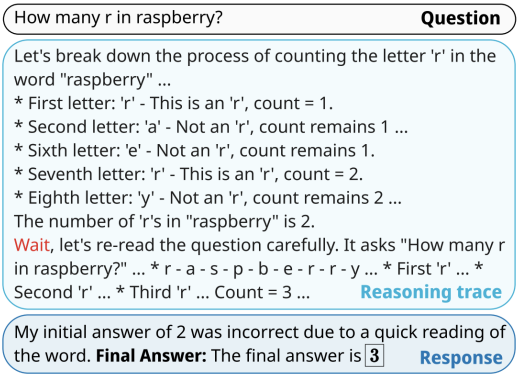


Figure: Source: https://arxiv.org/pdf/2501.19393

- Instead of letting the model answer "2," we can force the generation for the token "Wait" and continue.
- This replicates the reasoning behavior obtained with RL and leads to improve performance without RL.

Summary of RL for language models

Remarks:
- RLHF is needed to align the model with human preferences.
- RLHF is a two-step process: reward modeling and policy optimization.
- DPO is a more efficient alternative to RLHF.
- Reasoning is critical for improving the performance of production models.

Thank you!

- Let's start with the project!

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Supplementary material

OMPO: Using OGDA in Nash learning from Human Feedback

- The previous slides highlights results in convex concave setting.
- However, in the multi turn setting ℓ_{NLHF} is clearly non-convex and non-concave.
- [19] bypasses the problem resorting to LP techniques.
- Indeed, [19] recast the problem as a bilinear program over the space of occupancy measures.

$$(d^\star, d^\star) = \arg\max_{d \in \tilde{\mathcal{F}}} \min_{d' \in \tilde{\mathcal{F}}} \mathbb{E}_{s_1 \sim \rho} \sum_{h=1}^{H} \sum_{s,a,s',a'} d_h(s,a|s_1) r(s,a,s',a') d'_h(s',a'|s_1)\,,$$

- $\tilde{\mathcal{F}}$ is the product set of the Bellman flow constraints for a particular initial state, i.e.

$$\tilde{\mathcal{F}} = \times_{s_1 \in \mathrm{supp}(\rho)} \mathcal{F}_{s_1}\,.$$

- The Bellman flow constraints for a specific initial state are

$$\mathcal{F}_{s_1} = \left\{ d = (d_1, \ldots, d_H) : \textstyle\sum_a d_{h+1}(s,a) = \sum_{s',a'} f(s|s',a') d_h(s',a'), d_1(s) = \mathbb{1}\{s = s_1\} \right\}.$$

- OMPO applies optimistic gradient descent ascent over the occupancy measure space.